
2D-Shallow Water Equations

Basics – Solutions - Applications

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1 Basics

1.1 Equations and Variables

Variables	h = water depth [L] q = x- component of specific flow [$L^2 T^{-1}$] r = y- component of specific flow [$L^2 T^{-1}$]
Continuity-equation	$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial r}{\partial y} = 0$
Momentum-equations	$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h} + \frac{gh^2}{2} - \frac{h}{\rho} \tau_{xx} \right) + \frac{\partial}{\partial y} \left(\frac{qr}{h} - \frac{h}{\rho} \tau_{xy} \right) = -gh \frac{\partial z_b}{\partial x} - \frac{\tau_{bx}}{\rho}$ $\frac{\partial r}{\partial t} + \frac{\partial}{\partial x} \left(\frac{qr}{h} - \frac{h}{\rho} \tau_{xy} \right) + \frac{\partial}{\partial y} \left(\frac{r^2}{h} + \frac{gh^2}{2} - \frac{h}{\rho} \tau_{yy} \right) = -gh \frac{\partial z_b}{\partial y} - \frac{\tau_{by}}{\rho}$
Constants and derived quantities	z_b = bottom elevation [L], τ_{bi} = bottom shear stress [$M L^{-1} T^{-2}$], ρ = density of fluid [$M L^{-3}$], τ_{ii} = turbulent normal stresses, τ_{ij} = turbulent shear stresses
Bottom-shear stresses	$\frac{\tau_{bx}}{\rho} = \frac{u\sqrt{u^2 + v^2}}{c_f^2}; \quad \frac{\tau_{by}}{\rho} = \frac{v\sqrt{u^2 + v^2}}{c_f^2}$ <p>with (u,v) = x- and y-components of flow velocity and c_f = friction coefficient (e.g. from Manning-Strickler-Formula)</p>
Turbulent shear stresses	<ul style="list-style-type: none"> – Influence of turbulent shear stress is important, if velocity gradients are large (e.g. in separations, constrictions) – For many applications (e.g. dam break waves, flood waves) the influence of turbulent stresses can be neglected. – Overview over various approaches in: Rodi W. 1984. Turbulence models and their application in hydraulics. A state-of-the-art paper. IAHR, Delft

Table 1 - Overview 2D-Shallow Water Equations

1.2 Initial and Boundary Conditions

For the starting time of the simulation at any computational node the values of the 3 variables (h , q , r) must be given. If no suitable information is available, one of the following options is often chosen in practice:

- (1) Assumption of fluid at rest, constant hydraulic head $z = z_b + h = \text{const}$, $q = r = 0$.

- (2) Assumption of a dry channel („dry start“): $h = q = r = 0$.
- (3) Using results from an earlier simulation as initial conditions („hot start“)

Depending on the flow regime the variables must also be given at the model boundaries. The type of variables depends on the flow regime and is determined by the number of characteristics which arrive at the boundary from outside:

<i>Flow regime</i>	<i>Inflow bound.</i>	<i>Outflow bound.</i>
Subcritical	2	1
supercritical	3	0

Table 2 – Number of required boundary conditions for the 2D-Shallow water equations

Setting meaningful boundary conditions is done in analogy to the 1D case. At an inflow boundary the flow rate Q should be given as inflow hydrograph together with the direction of the inflow. For an outflow boundary the following quantities could be used:

- (1) Water table elevation (possibly hydrograph)
- (2) Water level-flow relationship (if known)
- (3) Slope of water surface or energy slope

Suggestion: A comparison with a physical model can simplify the choice of boundary conditions. Boundaries which can be realized in a physical model will as a rule also lead to reasonable results in a numerical model.

1.3 Validity

The shallow water equations are based on the assumption that over the flow depth the pressure distribution is hydrostatic. This is the case for long and shallow waves (i.e. waves with a wave length much larger than water depth), in which the vertical acceleration of fluid elements during the wave passage stays small. When short or high waves pass, these conditions are violated. In linear wave theory one distinguishes between long waves (shallow water) and short waves (deep water), which are different with respect to the movement of water particles (Table below and lecture notes chapter 7).

<i>Region</i>	<i>Shallow water</i>	<i>Transition</i>	<i>Deep water</i>
Wave length	$L/h > 20$	$2 < L/h < 20$	$L/h < 2$
Wave celerity	$c = \sqrt{gh}$	$c = \sqrt{\frac{gL}{2\pi} \tanh \frac{2\pi h}{L}}$	$c = \sqrt{\frac{gL}{2\pi}}$
Particle path	linear	elliptical	circular

Table 3 – Comparison of different linear waves with L = wave length and h = water depth (from Liggett J. A. 1994. Fluid Mechanics. McGraw-Hill Int. Editions)

During the passage of high waves, significant vertical accelerations can occur, leading to deviation from the hydrostatic pressure conditions, an essential precondition of the usage of the shallow water waves. For situations which do not deviate too much from hydrostatic conditions, the classical shallow water equations can be generalized. The „Boussinesq-Equations“ – named after the French researcher Boussinesq (1842-1929) – contain additional terms which take into account the curvature of the water surface. They correct the erroneous observation that high waves compared to shallow ones are slower and subject to larger diffusion.

Under steady state flow the condition of hydrostatic pressure distribution may be violated at discontinuities of the water surface (hydraulic jump, pressure surge) or the channel bottom (steps). The solution of shallow water equations leads to local differences from observations. In such cases there are also extensions to the shallow water equations, which as a rule lead to new additional equations.

The 2D approach must fail in applications where 3D effects (e.g. secondary flows) become essential. This happens for strong curvatures of the channel or where the internal turbulent shear stresses influence the flow field considerably. The exact limit up to which a 2D model is still applicable also depends on the accuracy requirements. There are models, which do not average over depth but rather use the vertical velocity term resulting from the continuity equation. Such 3D shallow water equations are used for example in the simulation of lakes (e.g. circulation due to wind stress) or in coastal flows.

2 Numerical solution

2.1 Standard methods

The following methods are applied in solving the 2D-shallow water equations: Finite-Difference-Method (FDM), Finite-Element-Method (FEM) and Finite-Volume-Method (FVM). Besides the FEM the following two methods have found a larger fan community:

<i>Staggered Grid Method (SGM)</i>	<i>Cell centered - FVM</i>
<ul style="list-style-type: none"> – Standard method since the 60s – implicit time discretization – very compact and efficient (Flux computation in parts without interpolation) – usually based on rectangular grids – stable only for subcritical regimes! 	<ul style="list-style-type: none"> – developed in Fluid Dynamics (from 1980) – explicit time discretisation – requires smooth grids (otherwise inconsistent) – stable for sub- and supercritical regimes

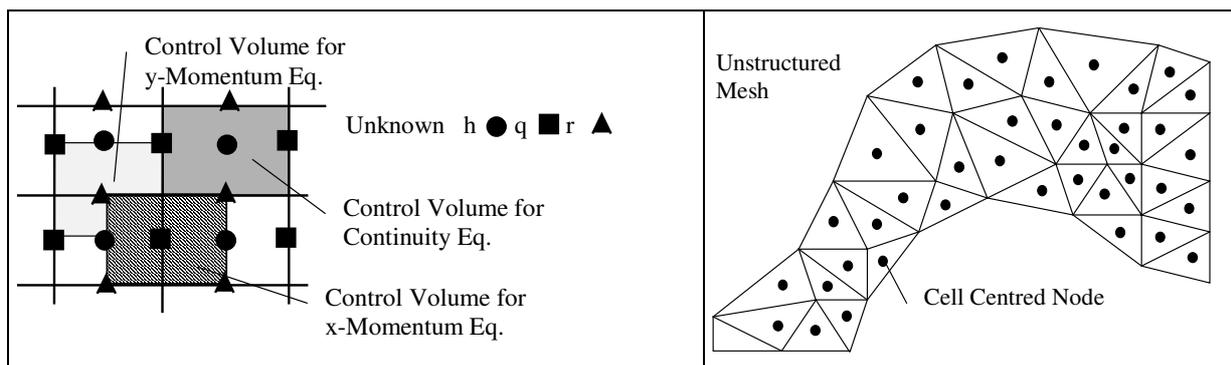


Table 4 – Comparison of two standard methods for the solution of the shallow water equations

The staggered grid method has been used since the 60s, above all in coastal flows. A weak point is the instability for supercritical flows.

The cell-centered FVM is stable for both sub- and supercritical flow regimes. Therefore this method is the favorite method applied today for inland flows.

Important Terms (explained for 1D case, but analogous in 2D case)

Consistency: Consistency means that the discrete equations approach (converge to) the differential equations for $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$.

Stability: A stable difference scheme prevents the unlimited growth of numerical error during calculation.

Convergence: The discrete solution U_i^n approaches the exact solution $U(x,t)$ of the differential equation at every point $x_i = i\Delta x$ and every time $t_n = n \Delta t$, if $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$.

Equivalence theorem by Lax: For a correctly posed initial value problem and a consistent discretization, stability is a necessary and sufficient condition for convergence.

2.2 Accuracy of the cell-centered Finite-Volume-Method (FVM)

The cell-centered FVM is of large practical importance due to its stability in both sub- and supercritical flows. To assess the accuracy of the method we look at the one-dimensional advection equation as a model equation.

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{1}$$

with u = vector of unknowns and f = flux vector. A discrete formulation of the cell-centered FVM with centrally averaged fluxes reads

$$\frac{\partial u_i}{\partial t} \Delta x_i + \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}(f_{i-1} + f_i) = 0 \tag{2}$$

The accuracy of the discrete formulation can be determined by local Taylor expansion of the fluxes around point i :

$$f_{i\pm 1} = f_i \pm \Delta x_{\pm} \frac{\partial f_i}{\partial x} + \frac{\Delta x_{\pm}^2}{2} \frac{\partial^2 f_i}{\partial x^2} \pm \frac{\Delta x_{\pm}^3}{6} \frac{\partial^3 f_i}{\partial x^3} + \dots \quad (3)$$

(for nomenclature see figure below).

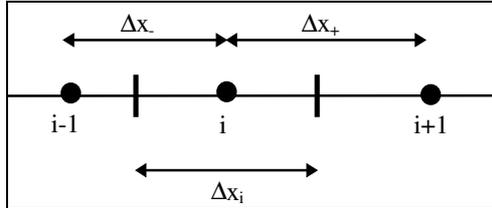


Figure 1 – Configuration of computational nodes and nomenclature

Insertion of expression (3) into (2) and division by Δx_i leads to the differential equation belonging to discrete equation (2)

$$\frac{\partial u_i}{\partial t} + \underbrace{\frac{\Delta x_+ + \Delta x_-}{2\Delta x_i}}_{\text{inkonsistent}} \frac{\partial f_i}{\partial x} = - \underbrace{\frac{\Delta x_+^2 - \Delta x_-^2}{4\Delta x_i}}_{O(\Delta x)} \frac{\partial^2 f_i}{\partial x^2} - \underbrace{\frac{\Delta x_+^3 + \Delta x_-^3}{12\Delta x_i}}_{O(\Delta x^2)} \frac{\partial^3 f_i}{\partial x^3} - \dots \quad (4)$$

Comparison of equations (1) and (4) shows:

- (1) For equidistant grids ($\Delta x_i = \Delta x_- = \Delta x_+$) the accuracy of the cell centered-FVM is of second order (Error term $O(\Delta x)$ vanishes).
- (2) For smooth grids with $\Delta x_{\pm} = \Delta x \pm O(\Delta x^2)$ the accuracy is also of second order.
- (3) For non-smooth grids the accuracy is only of first order.
- (4) For non-smooth grids the discretization is not consistent, as $\frac{\Delta x_+ + \Delta x_-}{2\Delta x_i} \neq 1$.

2.3 Stability of the cell-centered FVM

Equation (2) can also be written as

$$\frac{\partial u_i}{\partial t} \Delta x_i + \frac{1}{2}(f_{i+1} - f_{i-1}) = 0 \quad (5)$$

i.e. the flux at point i disappears. A determination of fluxes from the arithmetic average of neighboring values has the consequence that u_i becomes independent of f_i . In the results this shows in a numerical solution which is not smooth but has wiggles (compare figure below).

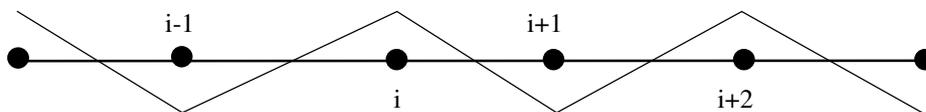


Figure 2: Wiggles in the discrete solution when using centrally averaged fluxes

These can be suppressed by so called upwind-methods which introduce an additional numerical diffusion into the scheme. The flux at cell boundary $i+1/2$ is then

$$f_{i+1/2} = \frac{1}{2}(f_l + f_r) - \alpha(u_r - u_l) \quad (6)$$

where the indices l and r indicate the values to the left and to the right of the cell boundary considered. To determine the factor α there are a number of different suggestions, which differ in computational effort and physical motivation (see lecture script chapter 6).

With the obvious choice $l = i$ and $r = i+1$ for the indices one obtains a scheme of first order accuracy. For applications, where numerical diffusion is undesirable (e.g. in the simulation of short waves) the accuracy of the methods can be increased. To do so, the flux terms are determined from the neighboring node values by extrapolation of variables. In order to guarantee a smooth behavior of the solution, even if there are discontinuities, the flux terms must be limited by so called limiter functions. At the location of a discontinuity (front) the accuracy is reduced by the limiter to first order.

Temporal discretization

The discretization in time can be formulated in different ways:

- Explicit methods compute the variables of the next time step exclusively from already known values of the variable at the previous time step. The category of explicit methods also contains the so called multiple step methods (e.g. predictor-corrector method). Explicit methods are in general easily programmed but have the restriction that they must fulfill a stability criterion (compare below).
- Implicit methods use in addition the values of hitherto unknown values of the variables at the new time. The solution in point i therefore also depends from the still unknown solution at the neighboring points of point i . This requires solving of an equations system (as opposed to a single equation in the explicit method) and – due to non-linear terms – eventually an iterative procedure. Implicit methods involve much more effort in programming. At the same time they allow to choose larger time steps.

Explicit methods gain in importance due to the progress in computer performance. They have to fulfill the Courant-Friedrich-Levy (CFL) condition. This CFL-number is a measure for the progress of a disturbance over a time step Δt related to the grid distance Δx

$$CFL = |\lambda| \frac{\Delta t}{\Delta x} = (|u| + c) \frac{\Delta t}{\Delta x} \quad (9)$$

with u = flow velocity and c = wave velocity. An explicit method is only stable for small CFL-numbers (usually $CFL < 1.0$) Equation (9) thus leads to an equation for the maximum permissible time step length at fixed grid size Δx (provided the largest λ is chosen). In 2D the condition is analogous.

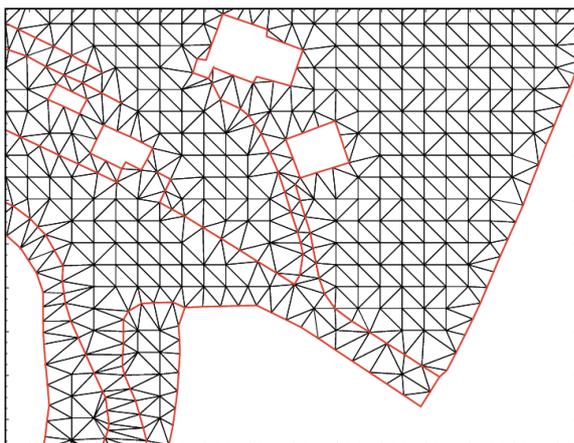
3 Model application

3.1 Grid generation

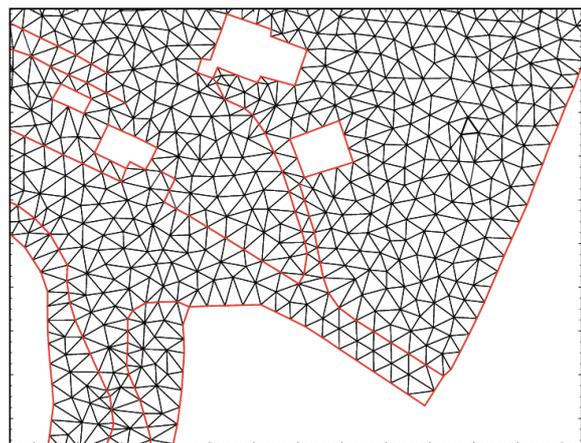
In the practical application of 2D models, the grid generation is a decisive step. Often it decides upon success or failure of a model, both with respect to the quality of the results as well as the time consumption for the user.

Application of the FVM requires „smooth“ grids. That means neighboring cells should have similar dimensions. When using triangles, the appearance of acute-angled triangles (i.e. triangles with one angle larger 90°) should be avoided. Digital terrain models on the basis of TINs (Triangulated Irregular Networks), which are the basis for the spatial discretization in 2D, often contain such triangles. That means the triangulated DTM is usually not directly utilizable as discretization grid.

A free tool for the construction of quality grids is the program „Triangle“ (<http://www.cs.cmu.edu/~quake/triangle.html>).



Grid generated by buffering of a point raster



Grid generated by quality refinement with Program Triangle

3.2 Model tests

As shown, the FVM are consistent on smooth grids. That means the discretization error decreases with increasing resolution of the grid. This property can (should) be used to check the model error. To do so, computations with grids of different refinement are performed and compared. If such solutions do not show any differences, we have reached grid convergence and the discretization error has no influence on the results. In the opposite case, the grid resolution has to be increased and the test has to be repeated.

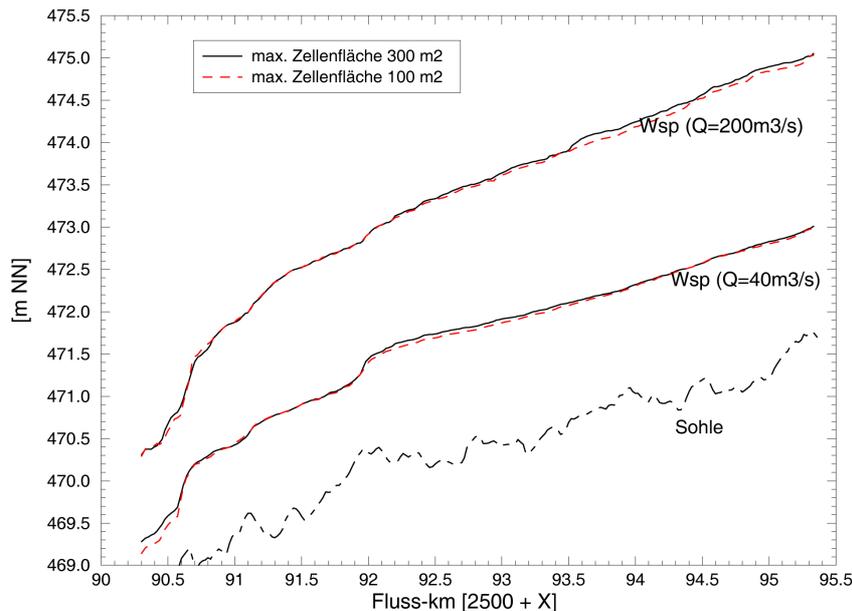


Figure 3 – Model test: Comparison of the computed water surface elevations for different grid refinement (Longitudinal profile of the Danube near Ulm)

3.3 Validation

Calibration: Adjustment of model parameters (e.g. friction coefficients), until results reproduce measurement data sufficiently well.

Inverse Modeling: Automatic calibration of model parameters by adjustment of parameters to reproduce the observed quantities (e.g. Levenberg-Marquardt-Algorithm)

Validation: Calculation of observed quantities, which have not been used in calibration (and without further change of parameters), and discussion of the differences (including critical discussion of the quality of the measured quantities).

Provocative hypothesis (Prof. Jean Cunge, University of Grenoble):

„By calibration simulation models lose their prognostic capability!“

Consequences for the application of flow models (Good Modeling Practice):

- (1) Do not overdo the fitting of parameters especially in the light of uncertain observations
- (2) Use observations primarily for validation (Learning effect!)

3.4 Areas of application

- (1) **Risk assessment:** Simulation of floods, dam and dike break simulation, hydraulic loads of hydraulic structures along and in the river
- (2) **Hydraulic engineering:** Computation of water surface elevations for flows; Design of protective structures, optimization of structures, abstractions, in- and outflows.

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- (3) Ecological questions: Assessment of constructive measures on flow (flow velocities, bottom shear stresses); Assessment of permeability for fish and properties of habitats, water abstractions, transport of solutes (pollutants) in the far field and intermediate field
- (4) Sediment balance: Bottom erosion, flushing, sedimentation of reservoirs

4 Control questions

4.1 Concerning the shallow water equations

- (1) Which are the unknowns in the shallow water equations?
- (2) Which quantities have to be found by empirical formulae?
- (3) Give number and types of boundary conditions for inflow and outflow boundaries.
- (4) Which assumptions limit the applicability of the shallow water equations?
- (5) What is the difference between shallow and deep water waves?
- (6) Give typical fields of application for the shallow water equations.

4.2 Concerning the numerical solution

- (1) What do the terms „consistency“, „stability“ and „convergence“ mean in the context of numerical solution methods?
- (2) Under which condition is the Finite-Volume-Method (FVM) consistent?
- (3) What is the difference between explicit and implicit solution methods (advantages and disadvantages)?
- (4) How is the CFL-number defined and what is its significance in the application of explicit solution methods?
- (5) How are smooth grids characterized and how can they be generated?
- (6) How can the size of the discretization error be estimated and how can it be eliminated?